MODELING, ANALYSIS, AND CONTROL OF DYNAMIC SYSTEMS

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CHAPTER ONE
Introduction

Modeling, analysis, and control of dynamic systems have been of interest to engineers for a long time. Within recent years the subject has increased in importance for three reasons. Before the invention of the digital computer, the calculations required for meaningful applications of the subject were often too time consuming and error prone to be seriously considered. Thus gross simplifications were made, and only the simplest models of transient behavior were used, if at all. Now, of course, the widespread availability of computers, as well as pocket calculators, allows us to consider more detailed models and more complex algorithms for analysis and design.

Second, with this increased computational power, engineers have correspondingly increased the performance specifications required of their designs to make better use of limited materials and energy, for example, or to improve safety. This leads to the need for more detailed models, especially with regard to the prediction of transient behavior.

Finally, the use of computers as system elements for measurement and control now allows more complex algorithms to be employed for data analysis and decision making. For example, intelligent instruments with microprocessors can now calibrate themselves. This increased capability requires a better understanding of dynamic systems so that the full potential of these devices can be realized.

1.1 SYSTEMS

The term *system* has become widely used today, and as a result, its original meaning has been somewhat diluted. *A system is a combination of elements intended to act together to accomplish an objective.* For example, an electrical resistor is an element for impeding the flow of current, and it usually is not considered to be a system in the sense of our definition. However, when it is used in a network with other resistors, capacitors, inductors, etc., it becomes part of a system. Similarly, a car's engine is a system whose elements are the carburetor, the ignition, the crankshaft, and so forth. On a higher level the car itself can be thought of as a system with the engine as an element. Since nothing in nature can be completely isolated from everything else, we see that our selection of the "boundaries" of the system depends on the purpose and the limitations of our study. This in part accounts for the widespread use of the term *system*, since almost everything can be considered a system at some level.

The Systems Approach

Automotive engineers interested in analyzing the car's overall performance would not have the need or the time to study in detail the design of the gear train. They most likely would need to know only its gear ratio. Given this information, they would then consider the gear train as a "black box." This term is used to convey the fact that the details of the gear train are not important to the study (or at least constitute a luxury they cannot afford). They would be satisfied as long as they could compute the torque and speed at the axle, given the torque and speed at the drive shaft.

The black-box concept is essential to what has been called the "systems approach" to problem solving. With this approach each element in the system is treated as a black box, and the analysis focuses on how the connections between the elements influence the overall behavior of the system. Its viewpoint implies a willingness to accept a less detailed description of the operation of the individual elements in order to achieve this overall understanding. This viewpoint can be applied to the study of either artificial or natural systems. It reflects the belief that the behavior of complex systems is made up of basic behavior patterns that are contributed by each element and that can be studied one at a time.

The behavior of a black-box element is specified by its *input-output relation.* An *input* is a *cause*; an *output* is an *effect* due to the input. Thus the input-output relation expresses the cause-and-effect behavior of the element. For example, a voltage v applied to a resistor R causes a current i to flow. The input-output or causal relation is $i = v/R$. Its input is v, its output is i, and its input-output relation is the preceding equation.

Block Diagrams

The black-box treatment of an element can be expressed graphically, as shown in Figure 1.1. The box represents the element, the arrow entering the box represents the input, and that leaving the box stands for the output. Inside the box we place the mathematical expression that relates the output to the input, if this expression is not too cumbersome, and is known. This graphical representation is a *block diagram.*

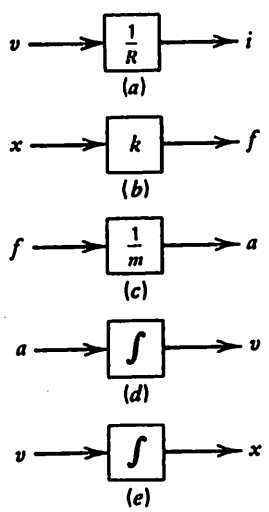


Figure 1.1 Block diagrams of input-output relations. (*a*) Voltage-current relation for a resistor. (*b*) Displacement-force relation for a spring. (*c*) Force-acceleration relation for a mass. (*d*) Velocity as the time integral of acceleration. (*e*) Displacement as the time integral of velocity.

The diagram in Figure 1.1*a* represents the resistor with input v and output i. They are related by the constant $1/R$. Figure 1.1*b* shows the representation of a spring whose resisting tensile force f is proportional to its extension x so that $f = kx$. Figure 1.1*c* shows how a force f applied to a mass m causes an acceleration a. The governing relation is Newton's law: $f = ma$. To obtain a from f, we must multiply the input f by the constant $1/m$. Thus the symbol in the box represents the operation that must be performed on the input to obtain the output.

Not all black-box representations must refer to actual physical elements. Since they express cause-and-effect relations they can be used to display processes as well as components. Two examples of this are shown in Figures 1.1*d* and 1.1*e*. If we integrate the acceleration a over time, we obtain the velocity v, that is, $v = \int a\,dt$. Thus acceleration is the cause of velocity. Similarly, integration of velocity produces displacement x: $x = \int v\,dt$. The integration operator within each box in Figures 1.1*d* and 1.1*e* expresses these facts. Whenever an output is the time integral of the input, the element is said to exhibit *integral causality*. We will see that integral causality constitutes a basic form of causality for all physical systems.

The input-output relations for each element provide a means of specifying the connections between the elements. When connected together to form a system, the inputs to some elements will be the outputs from other elements. For example, the position of a speedometer needle is caused by the car's speed. Thus for the speedometer element, the car's speed is an input. However, the speed is the result of action of the drive-train element. The input-output relation can sometimes be reversed for an element, but not always. We can apply a current as input to a resistor and consider the voltage drop to be the output. On the other hand, the position of the speedometer needle can in no way physically influence the speed of the car.

The system itself can have inputs and outputs. These are determined by the selection of the system's boundary. Any causes acting on the system from the world external to this boundary are considered to be system inputs. Similarly, a system's outputs can be the outputs from any one or more of the elements, viewed in particular from outside the system's boundary. If we take the car engine to be the system, a system input would be the throttle position determined by the acceleration pedal, and a system output would be the torque delivered to the drive shaft. If the car is taken to be the system instead, the input would still be the pedal position, but the outputs might be taken to be the car's position, velocity, and acceleration. Usually our choices for system outputs are a subset of the possible outputs and are the variables in which we are interested. For example, a performance analysis of the car would normally focus on the acceleration or velocity, but not on the car's position.

A simple example of a system diagram is provided in Figure 1.2. Suppose that a mass m is connected to one end of a spring. The other end of the spring is attached to a rigid support. In addition to the spring force f_s, another force f_o acts on the mass. This force is considered to be due to the external world and acts across the system "boundary"; that is, it is not generated by any action within the system itself. It might be due to gravity, for example.

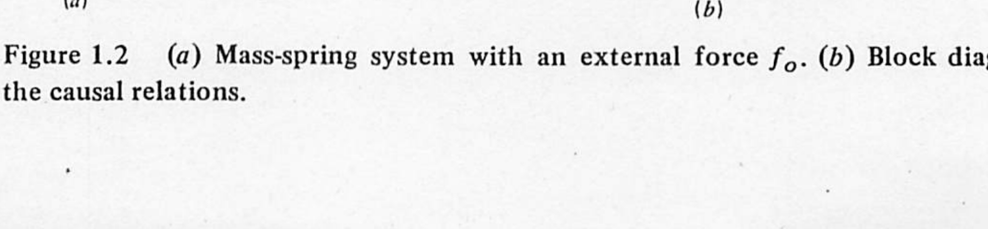


Figure 1.2 (*a*) **Mass-spring system with an external force f_o.** (*b*) **Block diagram of the causal relations.**

The cause-and-effect relations can be summarized by the system diagram in Figure 1.2*b*. The net force on the mass in the direction of positive displacement x is $f_o - f_s$, since the spring will pull up on the mass if the mass position is below the rest position ($x > 0$). The addition and subtraction of forces to produce the net force are represented by a new symbol, the *comparator* – a circle whose output is the signed sum of the inputs. The plus (+) sign indicates that f_o is to be added; the minus (−) sign indicates that f_s is to be subtracted to produce the net force.

The system input is f_o; its output could be any or all of the variables generated within the diagram. If we are interested only in the displacement x, then its arrow is shown leaving the system.

System diagrams such as this are a visually concise summary of the interplay between the causes and the effects. We will use them often.

Static and Dynamic Systems

In general the present value of an element's output is the result of what has happened to the element in the past as well as what is currently affecting it. For example, the present position of a car depends on where it started and what its velocity has been from the start. We define a *dynamic* element to be one whose present output depends on past inputs. Conversely, a *static* element is one whose output at any given time depends only on the input at that time.

For the car considered as an element with an acceleration pedal position as input and car position as output, the preceding definition shows the element to be dynamic. On the other hand, we can consider a resistor to be a static element because its present current depends only on the voltage applied at present, not on past voltages. This is an approximation, of course, because the resistor cannot respond instantaneously to voltage changes. This is true of all physical elements, and we therefore conclude that a static element is an approximation. Nevertheless, it is widely used because it results in a simpler mathematical representation.

In popular usage the terms *static* and *dynamic* are used to distinguish situations in which no change occurs from those that are subject to changes over time. This usage conforms to the preceding definitions of these terms if the proper interpretation is made. A static element's output can change with time only if the input changes and will not change if the input is constant or absent. However, if the input is constant or is removed from a dynamic element, its output can still change. For example, if the car's engine is turned off, the car's position will continue to change because of the car's velocity (because of past inputs). A similar statement cannot be made for the electrical resistor.

In the same way we also speak of static and dynamic systems. A static system contains all static elements. Any system that contains at least one dynamic element is a dynamic system.

1.2 MODELING, ANALYSIS, AND CONTROL

We live in a universe that is undergoing continual change. This change is not always apparent if its time scale is long enough, such as with some geologic processes, but as engineers we often must deal with situations in which time-dependent effects are important. For example, design of high-speed production machinery for precise operation requires that the vibrational motions due to high accelerations be small in amplitude and die out quickly. Likewise, time-dependent behavior of fluid–flow and heat-transfer processes significantly affects the quality of the product of a chemical process. Even a relatively "stationary" object like a bridge must be designed to accommodate the motions and forces produced by a heavy rolling load.

Modeling

In order to deal in a systematic and efficient way with problems involving time-dependent behavior, we must have a description of the objects or processes involved. We call such a description a *model.* A model for enhancing our understanding of the problem can take several forms. A physical model, like a scale model, helps us to visualize how the components of the design fit together and can provide insight not obtainable from a blueprint (which is another model form). Graphs or plots are still another type of model. They can often present time-dependent behavior in a concise way, and for that reason we will rely heavily on them throughout this study. The model type we will use most frequently is the *mathematical model*, which is a description in terms of mathematical relations. These relations will consist of differential or difference equations if the model is to describe a dynamic system.

The concept of a mathematical model is undoubtedly familiar from elementary physics. Common examples include the voltage-current relation for a resistor $v = iR$, and the force-deflection relation for a spring $f = kx$. One of our aims here is to introduce a framework that allows the development of mathematical models for describing the time-dependent behavior of many types of phenomena: fluid flow, thermal processes, mechanical elements, and electrical systems, as well as some nonphysical applications. In this regard, it is important to remember that the precise nature of a mathematical model depends on its purpose. For example, an electrical resistor can be subjected to mechanical deformations if its mounting board is subjected to vibration. In this case, the force-deflection spring model could be used to describe the resistor's mechanical behavior.

Thus we see that the nature of any object has many facets: thermal, mechanical, electrical, etc. No mathematical model can deal with all these facets. Even if it could, it would be useless because its very complexity would render it cumbersome. We can make an analogy with maps. A given region can be described by a road map, a terrain-elevation map, a mineral-resources map, a population-density map, and so on. A single map containing all this information would be cluttered and useless. Instead, we select the particular type of map required for the purpose at hand. In the same way, we select or construct a mathematical model to suit the requirements of the study.

The purpose of the model should guide the selection of the model's time scale, its length scale, and the particular facet of the object's nature to be described (thermal, mechanical, electrical, etc.). The time scale will in turn determine whether or not time-dependent effects should be included. (Tectonic plate motion constitutes time-dependent behavior on a geologic time scale but would not be considered by an engineer designing a bridge.) Similarly, the length scale partly dictates what details should or should not be included. The engineer analyzing the dynamics of high-speed machinery might treat a component as a point mass, whereas this approximation is useless to a metallurgist studying material properties at the molecular level.

Block diagrams are often used to display the mathematical model in a form that allows us to understand the interactions occurring between the system's elements. For example, the mathematical model of the mass-spring system shown in Figure 1.2*a* is

$$f_o - f_s = ma$$

$$f_s = kx$$

$$x = \int v\,dt$$

$$v = \int a\,dt$$

Each equation is a cause-and-effect relation for one part of the system.

The diagram is a valuable aid, but if we wish to solve for the displacement $x(t)$, we need the model in equation form. If we differentiate the last two relations we obtain

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = a$$

or

$$a = \frac{d^2x}{dt^2}$$

Substitution of this and the second relation into the first gives

$$f_o - kx = m\frac{d^2x}{dt^2} \tag{1.2-1}$$

This differential equation is a quantitative description of the system. Given f_o, k, m, and the initial position $x(0)$ and velocity $v(0)$, we can use the methods of later chapters to solve the equation for $x(t)$.

Analysis

A mathematical model represents a concise statement of our hypotheses concerning the behavior of the system under study. We can deal with the verification of the model in two ways. Verification by experiment or testing is ultimately required of all serious design projects. This is not always done at the outset of a study, however, especially if one is dealing with component types whose behavior is known to be well described by a specific model on the basis of past experience. For example, we are on firm ground in using the resistor model $v = iR$ without verification, as long as the operating conditions (voltage levels, temperatures, etc.) are not extreme. This is often the case for the types of problems we will be considering, since we will be concerned with the behavior of systems consisting of components whose individual behavior is often well understood.

Once we are satisfied with the validity of our chosen component models, they can be used to predict the performance of the system in question. Predicting the performance from a model is called *analysis*. For example, the current produced in a resistor by an applied voltage v can be predicted to be $i = v/R$ by solving the resistor model for the unknown variable i in terms of the given quantities v and R. Most of our mathematical models will describe dynamic behavior and thus will consist of differential or difference equations. They will not be as easy to solve as the algebraic model just seen. Nevertheless, the techniques for analyzing such models are straightforward. These are introduced in Chapter Three.

Just as the model's purpose partly determines its form, so also does the purpose influence the types of analytical techniques used to predict the system's behavior. It is not possible to discuss these concepts with the simple resistor model, because its solution is so simple. However, we will be developing many types of analytical techniques whose applicability depends on the purpose of the analysis. Not all of these techniques will be brought to bear on any one problem, but the engineer should be familiar with all of them. They are the tools of the trade – a means to an end. We will not study them, as a mathematician would, for their inherent interest. Instead, we will focus on how they can help us predict the performance of a proposed design before it is built. Thus we can avoid a cut-and-try approach. This is especially important today since most modern engineering endeavors are too complex and expensive to allow them to be built without a thorough analysis beforehand.

Control

The successful operation of a system under changing conditions often requires a control system. For example, a building's heating system requires a thermostat to turn the heating elements on or off as the room temperature rises and falls (Figure 1.3). Note that we have not shown the thermostat as one element because it has two functions: (1) to measure the room temperature and compare it with the desired temperature and (2) to decide whether to turn the furnace on or off. The variation in the outdoor environment is the primary reason for the unpredictable change in the room temperature. If the outside conditions (temperature, wind, solar insolation, etc.) were predictable, we could design a heater that would operate continuously to supply heat at a predetermined rate just large enough to replace the heat lost to the outside environment. No controller would be necessary. Of course, the real world does not behave so nicely, so we must adjust the heat-output rate of our system according to what the actual room temperature is.

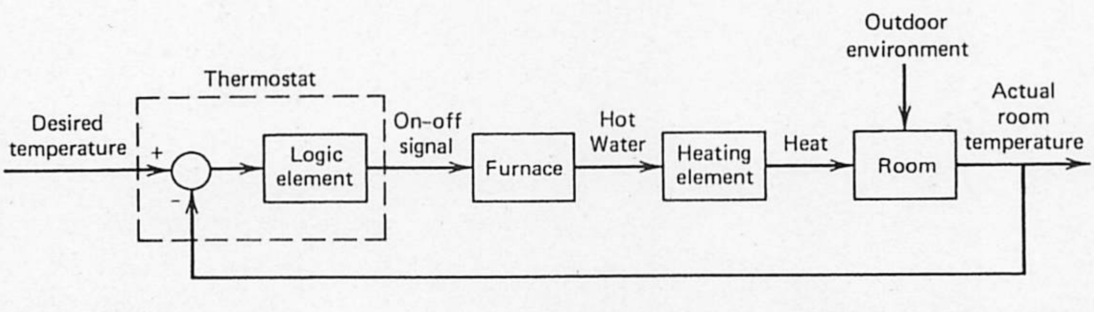


Figure 1.3 Block diagram of the thermostat system for temperature control.

We often wish to alter the desired operating conditions of a system. In our heating example the thermostat allows the user to specify the desired room temperature, say 68°F during the day and 60°F at night. When the thermostat setting is changed from 60°F to 68°F in the morning, the thermostat acts to bring the room temperature up to 68°F and to keep it near this value until the setting is changed again at night.

The term *control* refers to the process of deliberately influencing the behavior of an object so as to produce some desired result. The physical device inserted for this purpose is the *controller* or *control system.* Other common examples of controllers include:

1. An aircraft autopilot for maintaining desired altitude, orientation, and speed.
2. An automatic cruise-control system for a car.
3. A pressure regulator for keeping constant pressure in a water-supply system.

A cutaway view of a commonly used type of pressure regulator is shown in Figure 1.4 along with a block diagram of its operation. The desired pressure is set by turning a calibrated screw. This compresses the spring and sets up a force that opposes the upward motion of the diaphragm. The bottom side of the diaphragm is exposed to the water pressure that is to be controlled. Thus the motion of the diaphragm is an

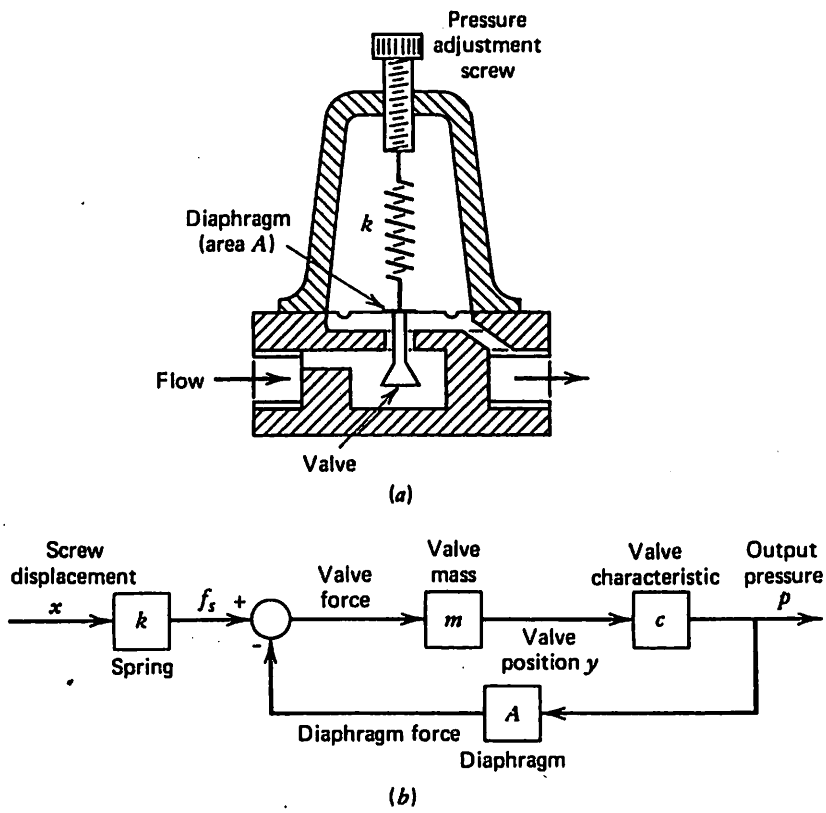


Figure 1.4 Pressure regulator. (*a*) Cutaway view. (*b*) Block diagram.

indication of the pressure difference between the desired and the actual pressures. It acts like a comparator. The valve is connected to the diaphragm and moves according to the pressure difference until it reaches a position in which the difference is zero.

From the preceding examples we see that the role of a controller is twofold.

1. It must bring the system's operating condition to the desired value.
2. It must maintain the desired condition in the presence of variations caused by the external environment.

In the terminology of the control engineer, we say that the controller must respond satisfactorily to changes in *commands* and maintain system performance in the presence of *disturbances.*

One or more controllers are often required in complex dynamic systems in order to make the system elements act together to achieve the intended goal. So the design of dynamic systems quite naturally involves the study of control systems. On the other hand, the variations produced by command changes and disturbances tend to upset the system. Thus control system design requires models that describe the dominant dynamic properties of the system to be controlled, and the analysis techniques must be capable of dealing with such a model. Modeling, analysis, and control of dynamic systems therefore constitute a unified area of study.

1.3 TYPES OF MODELS

As we have seen, a system model is a representation of the essential behavior of the system for the purposes at hand. In order to be useful it must contain the minimum amount of information necessary to achieve its purpose, and no more. This requirement is most immediately reflected in the choice of static- versus dynamic-element models. Those elements whose behavior is fast relative to other elements are often modeled as static elements in order to reduce the complexity of the model. For example, the switching time of a thermostat is fast compared to the time required for the room temperature to change appreciably. Thus the room temperature in Figure 1.2 would most likely be modeled as a dynamic element and the thermostat as a static one.

Lumped and Distributed-Parameter Models

We have implicitly assumed in the heating example that the temperature in the room can be described by a single number, a temperature that is average in some sense. In reality the temperature varies according to location within the room, but if we did not choose to use a single representative temperature the required room model would be much more complicated.

Many variables in nature are functions of location as well as time. The process of ignoring the spatial dependence by choosing a single representative value is called *lumping* (room air is considered to be one "lump" with a single temperature).

Lumping an element is a technique usually requiring experience. It reflects the judgment of the engineer as to what is unimportant in terms of spatial variation.

It can be described as the spatial equivalent of the process of dividing a system into static and dynamic elements. The model of a lumped element or system is called a *lumped-parameter model.* If it is dynamic, the only independent variable in the model will be time; that is, the model will be an ordinary differential equation, like (1.2-1). Only time derivatives will appear, not spatial derivatives.

When spatial dependence is included, the independent variables are the spatial coordinates as well as time. The resulting model is said to be a *distributed-parameter model.* It consists of one or more partial differential equations containing partial derivatives with respect to the independent variables. The difference is illustrated in Figure 1.5*a*, which shows the temperature T of a metal plate. If the plate is heated at one side, the temperature will be a function of location and time – $T = T(t, x, y, z)$ – and the model will be of the form

$$f\left(T, \frac{\partial T}{\partial t}, \frac{\partial^2 T}{\partial x^2}, \frac{\partial^2 T}{\partial y^2}, \frac{\partial^2 T}{\partial z^2}\right) = 0 \tag{1.3-1}$$

But if the plate temperature is lumped with a single value, the model will be of the form

$$f\left(T, \frac{dT}{dt}\right) = 0 \tag{1.3-2}$$

which is easier to handle mathematically.

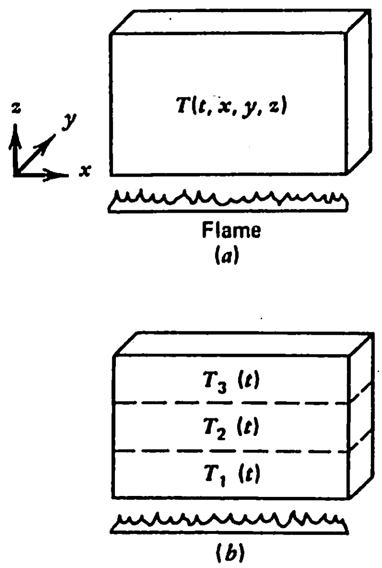


Figure 1.5 Temperature distribution in a plate. (*a*) Distributed-parameter representation. (*b*) Lumped-parameter representation using three elements.

Lumping may be done at several levels. For example, we may take a single temperature to represent the entire house. In this case a single differential equation would result. On the other hand, we may take a representative temperature for each room. In this case the total model would consist of a differential equation for each room temperature.

Although choosing a temperature for each room leads to several equations, the model is usually more manageable than if the lumping were not performed. .

There are applications in engineering where a detailed model like (1.3-1) is required, and we do not dismiss such models as useless. However, we will see that in analyzing systems with many elements, distributed-parameter models of elements are a luxury we usually cannot afford because their complexity tends to prevent us from understanding the overall system behavior. We therefore will limit our treatment to lumped-parameter models. Note that if a more detailed model is needed, we can increase the number of lumped elements, such as is shown in Figure 1.5*b*, where we have used three temperature lumps in an attempt to model the dependence of the plate temperature as a function of distance from the flame. The resulting model would have three coupled ordinary differential equations, one for each $T_i(t)$.

Linear and Nonlinear Models

We have seen that engineers should attempt to model elements as static rather than dynamic, and as lumped rather than distributed. The reason is that engineers eventually have to analyze the resulting system model, and its complexity can easily get out of hand if there is too much detail in each element model. In a similar vein we now discuss the distinction between model types based on *linearity*.

Let y be the output and x be the input of an element that can be either static or dynamic. Its model is written as

$$y = f(x) \tag{1.3-3}$$

where the function $f(x)$ may include operations like differentiation and integration. The model (or element) is said to be *linear* if, for an input $ax_1 + bx_2$, the output is

$$y = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = ay_1 + by_2 \tag{1.3-4}$$

where a and b are arbitrary constants, x_1 and x_2 are arbitrary inputs, and

$$y_1 = f(x_1) \tag{1.3-5}$$

$$y_2 = f(x_2) \tag{1.3-6}$$

Thus linearity implies that multiplicative constants and additive operations in the input can be factored out when considering the effects on the output. The linearity property (1.3-4) is sometimes called the *superposition principle* because it states that a linear combination of inputs produces an output that is the superposition (linear combination) of the outputs that would be produced if each input term were applied separately. Any relation not satisfying (1.3-4) is nonlinear.

Let us consider some input-output relations to see if they are linear. The simple multiplicative relation $y = mx$ is linear because

$$y = m(ax_1 + bx_2) = amx_1 + bmx_2 = ay_1 + by_2$$

where $y_1 = mx_1$ and $y_2 = mx_2$. The operation of differentiation $y = dx/dt$ is linear because

$$y = \frac{d}{dt}(ax_1 + bx_2) = a\frac{dx_1}{dt} + b\frac{dx_2}{dt} = ay_1 + by_2$$

Similarly, integration is a linear operation. If $y = \int x\,dt$, then

$$y = \int (ax_1 + bx_2)\,dt = a\int x_1\,dt + b\int x_2\,dt = ay_1 + by_2$$

Any relation involving a transcendental function or a power other than unity is nonlinear. For example, if $y = x^2$,

$$y = (ax_1^2 + bx_2^2) = a^2x_1^2 + 2abx_1x_2 + b^2x_2^2 \neq ax_1^2 + bx_2^2$$

Similarly, if $y = \sin x$,

$$y = \sin(ax_1 + bx_2) \neq a\sin x_1 + b\sin x_2$$

The definition of linearity (1.3-4) can be extended to include functions of more than one variable, such as $f(x, z)$. This function is linear if and only if

$$f(ax_1 + bx_2, az_1 + bz_2) = af(x_1, z_1) + bf(x_2, z_2)$$

Differential equations represent input-output relations also, and can be classified as linear or nonlinear. The outputs (solutions) of the model depend on the outputs' initial values and on the inputs. We will see in Chapter Three that a superposition principle applies to linear differential equations. This is useful because it allows us to separate the effects of more than one input and thus to consider each input one at a time. It also allows us to separate the effects of the initial values of the outputs from the effects of the inputs. For these reasons we will always attempt to obtain a linear model for our systems provided that any approximations required to do so do not mask important features of the system's behavior.

A differential equation is easily recognized as nonlinear if it contains powers or transcendental functions of the dependent variable. For example, the following equation is nonlinear.

$$\frac{dy}{dt} = -\sqrt{y} + f$$

Time-Variant Models

The presence of a time-varying coefficient does not make a model nonlinear. For example, the model

$$\frac{dy}{dt} = c(t)y + f$$

is linear. Models with constant coefficients are called *time-invariant* or *stationary* models, while those with variable coefficients are *time variant* or *nonstationary*. An example occurs if the mass m in Figure 1.1 represents a bucket of water with a leak. Its mass would then change with time, and $m = m(t)$ in (1.2-1).

Discrete and Continuous-Time Models

Sometimes it is inconvenient to view the system's dynamics in terms of a continuous-time variable. In such cases we use a discrete variable to measure time. Common examples of this usage include one's age (we usually express it in integer years, with no fractions) and interest computations on savings accounts (compounded quarterly, annually, etc.). For engineers the most important situation suggesting the use of discrete-time models occurs when a system contains a digital computer for measurement or control purposes. It is an inherently discrete-time device because it is driven by an internal clock that allows activity to take place only at fixed intervals. Thus a digital computer cannot take measurements continuously but must "sample" the measured variable at these instants.

If we choose to represent our system in terms of discrete time, the form of the model is a difference equation instead of a differential equation. For example, an amount of money x in a savings account drawing 5% interest compounded annually will grow according to the relation

$$x(k + 1) = 1.05x(k) \qquad (1.3\text{-}7)$$

The index k represents the number of years after the start of the investment. We will return to the analysis of discrete-time models in Chapter Four.

Model Order

Equation (1.2-1) is called a *second-order* differential equation because its highest derivative is second order. It is equivalent to the relations

$$m\frac{dv}{dt} = f_o - kx \tag{1.3-8}$$

$$\frac{dx}{dt} = v \tag{1.3-9}$$

These two first-order equations are coupled to each other because of the x term in the first equation and the v term in the second. One cannot be solved without solving the other; they must be solved simultaneously. Taken together they thus form a second-order model.

We will organize our study of dynamic systems partly according to the order of the model. Chapters Three and Four treat first-order continuous- and discrete-time models, while Chapter Five covers higher-order models. In this way we can begin simply and gradually progress to more difficult topics.

Stochastic Models

Sometimes there is uncertainty in the values of the model's coefficients or inputs. If this uncertainty is great enough, it might justify using a *stochastic* model. In such a model the coefficients and inputs would be described in terms of probability distributions involving, for example, their means and variances. Such a model would be useful for describing the effects of wind gusts on an aircraft autopilot. Although the wind is not random, presumably our knowledge of its behavior is poor enough to justify a probabilistic approach. However, the mathematics required to analyze such models is beyond the scope of this work, and we will not consider stochastic models further.

A Model Classification Tree

Figure 1.6 is a diagram of the relationship between the various model types. We have extended only the branches that lead to linear time-invariant models since this is the type of most interest to us.

1.4 LINEARIZATION

Because of the usefulness of the superposition principle, we always attempt to obtain a linear model if possible. Sometimes this can be done from the outset of neglecting effects that would lead to a nonlinear model. A common example of this is the small angle approximation. If we assume that the angle of rotation θ of the lever in Figure 1.7 is small, the rectilinear displacement of its ends is roughly proportional to θ such that $x = L\theta$. The same is not true for a large enough value of θ.

If such an approximation is not obvious, a systematic procedure based on the Taylor series expansion can be used (Appendix A). Let the input-output model for a

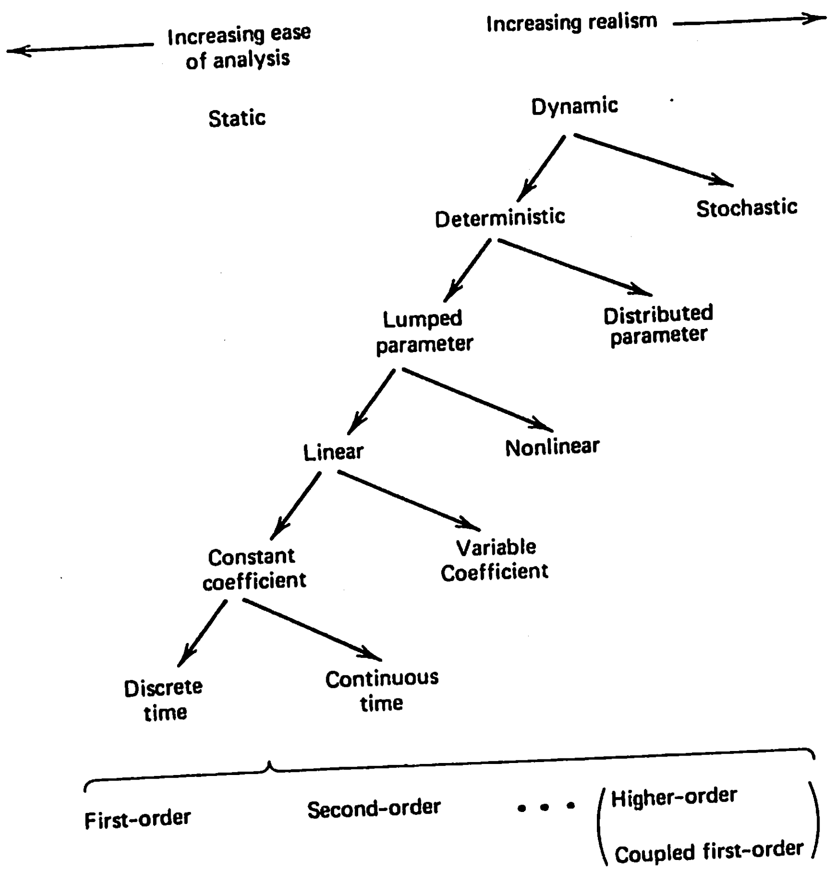


Figure 1.6 Classification of mathematical models. We have not continued the tree on every branch for simplicity. For example, nonlinear models can also be classified as discrete or continuous time.

static element be written as

$$w = f(y) \tag{1.4-1}$$

Its form is sketched in a general way in Figure 1.8. A model that is approximately linear near the reference point (w_o, y_o) can be obtained by expanding $f(y)$ in a Taylor series near this point and truncating the series beyond the first-order term. The series is

$$w = f(y) = f(y_o) + \left(\frac{df}{dy}\right)_o (y - y_o) + \frac{1}{2!}\left(\frac{d^2f}{dy^2}\right)_o (y - y_o)^2 + \dots \tag{1.4-2}$$

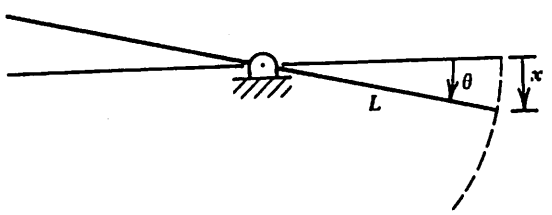


Figure 1.7 Small-angle approximation for the displacement of a lever endpoint. For θ small, $x \cong L\theta$.

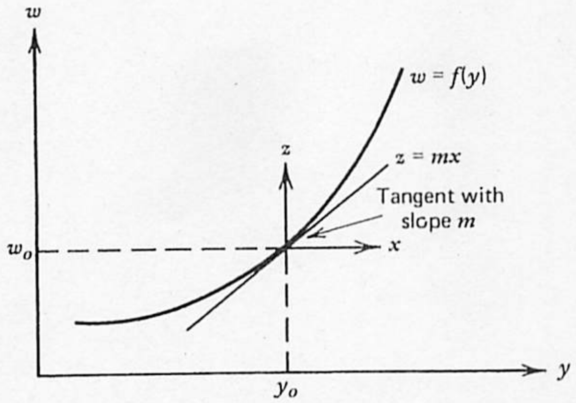


Figure 1.8 Linearization of the function $w = f(y)$ about the point (w_o, y_o).

where the subscript "o" on the derivatives means that they are evaluated at the reference point (w_o, y_o). If y is "close enough" to y_o, the terms involving $(y - y_o)^i$ for $i \geqslant 2$ are small compared to the first two terms. Ignoring these higher-order terms gives

$$w = f(y) \cong f(y_o) + \left(\frac{df}{dy}\right)_o (y - y_o) \tag{1.4-3}$$

This is a linear relation. To put it into a simpler form let

$$m = \left(\frac{df}{dy}\right)_o \tag{1.4-4}$$

$$z = w - w_o = w - f(y_o) \tag{1.4-5}$$

$$x = y - y_o \tag{1.4-6}$$

Then (1.4-3) becomes

$$z \cong mx \tag{1.4-7}$$

The geometric interpretation of this result is shown in Figure 1.8. We have replaced the original function with a straight line passing through the point (w_o, y_o) and having a slope equal to the slope of $f(y)$ at the reference point. With the (z, x) coordinates, a zero intercept occurs and the relation is simplified.

A Nonlinear Spring Example

No spring is linear over an arbitrary range of extensions. Instead, the force will increase nonlinearly with extension beyond some point, and the linear model used to obtain (1.2-1) will no longer be valid. Suppose the correct relation for a particular spring is $f = y^2$, where y is the extension of the spring from its free length (Figure 1.9). Let it be attached to the mass as shown in Figure 1.2 and allow the mass to settle to its rest position y_o. At this position the weight of the mass will equal the spring force so that $mg = y_o^2$, or $y_o = \sqrt{mg}$. The Taylor series applied to the spring relation $f = y^2$ gives

$$f \cong y^2 = y_o^2 + \left(\frac{dy^2}{dy}\right)_o (y - y_o)$$

$$= y_o^2 + 2y_o(y - y_o)$$

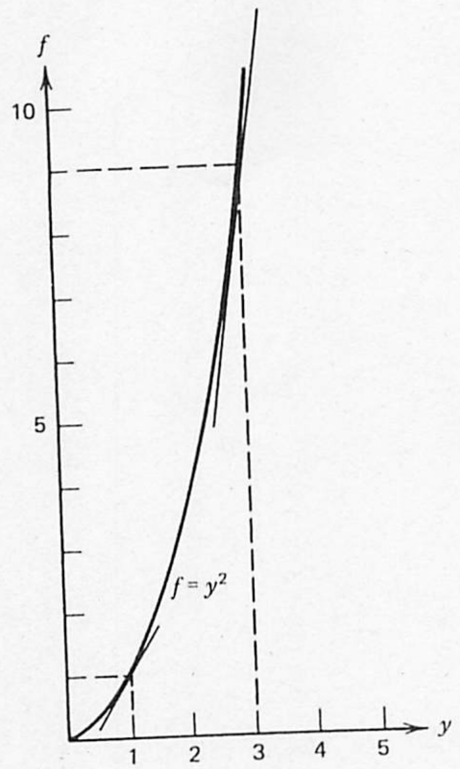


Figure 1.9 Linearization of the function $f = y^2$ about the points (1, 1) and (9, 3). Note the different slopes for each linearization.

Let $z = f - f_o = f - y_o^2$ and $x = y - y_o$. Thus

$$z \cong 2y_o x \tag{1.4-8}$$

The variable z is the change in spring force from its value at the rest position y_o. Equation (1.4-8) shows that the force change is approximately linear if y is close to y_o; that is, if x is small.

The free-body diagram of the mass is shown in Figure 1.10, with the gravity force and spring force being the only forces applied to the mass. From Newton's law,

$$m\frac{d^2x}{dt^2} = mg - (f_o + z)$$

Using (1.4-8) and the fact that $mg = y_o^2 = f_o$, we obtain

$$m\frac{d^2x}{dt^2} = -2y_o x \tag{1.4-9}$$

which is a linear equation. If we had not approximated the spring force as a linear function, the system's differential equation would have been

$$m\frac{d^2y}{dt^2} = mg - y^2 \tag{1.4-10}$$

which is nonlinear because of the y^2 term.

From (1.4-8) we see that the "spring constant" k is $2y_o$. It depends not only on the spring's physical properties, which yield the constant 2, but also on the reference position y_o. The constant $2y_o$ is the slope of the force-extension curve of the spring at the position y_o. This position is determined by the spring's characteristics and the weight of the mass. If $mg = 1$, then $y_o = 1$ and $k = 2$. For a larger weight, say $mg = 9$, $y_o = 3$ and $k = 6$.

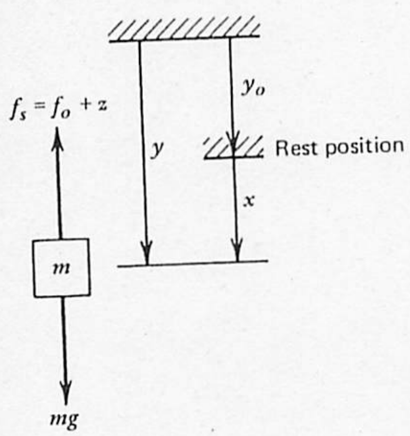


Figure 1.10 Free-body diagram of the mass-spring system.

The Multivariable Case

The Taylor series linearization technique can be extended to any number of variables. For two variables the function is

$$w = f(y_1, y_2) \tag{1.4-11}$$

and the truncated series is

$$w \cong f(y_{1o}, y_{2o}) + \left(\frac{\partial f}{\partial y_1}\right)_o (y_1 - y_{1o}) + \left(\frac{\partial f}{\partial y_2}\right)_o (y_2 - y_{2o}) \tag{1.4-12}$$

Define

$$z = w - w_o = w - f(y_{1o}, y_{2o}) \tag{1.4-13}$$

$$x_1 = y_1 - y_{1o} \tag{1.4-14}$$

$$x_2 = y_2 - y_{2o} \tag{1.4-15}$$

The linearized approximation is

$$z \cong \left(\frac{\partial f}{\partial y_1}\right)_o x_1 + \left(\frac{\partial f}{\partial y_2}\right)_o x_2 \tag{1.4-16}$$

The partial derivatives are the slopes of the function f in the y_1 and y_2 directions at the reference point.

As an example, consider the perfect gas law

$$p = \frac{mRT}{V} \tag{1.4-17}$$

where p, V, T, and m are the gas pressure, volume, temperature, and mass, respectively. The universal gas constant is R. If the gas is isolated in a flexible chamber, its mass is constant, but its volume, temperature, and pressure can change. For given reference values T_o and V_o, a linearized expression for the pressure is

$$p \cong p_o + \left(\frac{\partial p}{\partial T}\right)_o (T - T_o) + \left(\frac{\partial p}{\partial V}\right)_o (V - V_o) \tag{1.4-18}$$

where $p_o = mRT_o/V_o$ and

$$\left(\frac{\partial p}{\partial T}\right)_o = \left(\frac{mR}{V}\right)_o = \frac{mR}{V_o} = a \tag{1.4-19}$$

$$\left(\frac{\partial p}{\partial V}\right)_o = \left(-\frac{mRT}{V^2}\right)_o = -\frac{mRT_o}{V_o^2} = -b \tag{1.4-20}$$

Sometimes the following notation is used to represent the variations from the reference values.

$$\delta p = p - p_o$$

$$\delta T = T - T_o$$

$$\delta V = V - V_o$$

In this case (1.4-18) through (1.4-20) give

$$\delta p \cong a\,\delta T - b\,\delta V \tag{1.4-21}$$

Since a and b are positive, (1.4-21) shows that the pressure increases if T increases or if V decreases.

Of course, we do not need the linearized form to calculate p given V, T, m, and R. However, if p were to appear in a differential equation with T or V as inputs or dependent variables, then the linearized form would be needed to obtain a linear differential equation.

Linearization of Operating Curves

An element's input-output relation is not always given in analytical form, but might be available as an experimentally determined plot. For example, the operating curves of an electric motor might look something like those in Figure 1.11. For a fixed motor voltage v_2 and a fixed load torque T_2, the motor will eventually reach a fixed speed ω_2 some time after the motor is started. This steady-state speed can be found from the curve marked v_2. For another load torque, say T_a, the resulting speed ω_a can be found from the same v_2 curve. However, if we fix the load torque but change the voltage, the resulting speed must be found by interpolating between the appropriate constant-voltage curves.

Suppose that the load torque T and the voltage v are to be inputs for a dynamic model that will have the speed ω as the output. Then we cannot use the curves as is but must convert them to an analytical expression. This expression must be linear if the dynamic model is to be linear.

A linearized expression can be obtained from the operating curves by using (1.4-16) and calculating the required slopes numerically from the plot. The partial derivative $(\partial f/\partial y_1)_o$ is computed with y_2 held constant at the value y_{2o}. It can be found approximately as follows (see Figure 1.12).

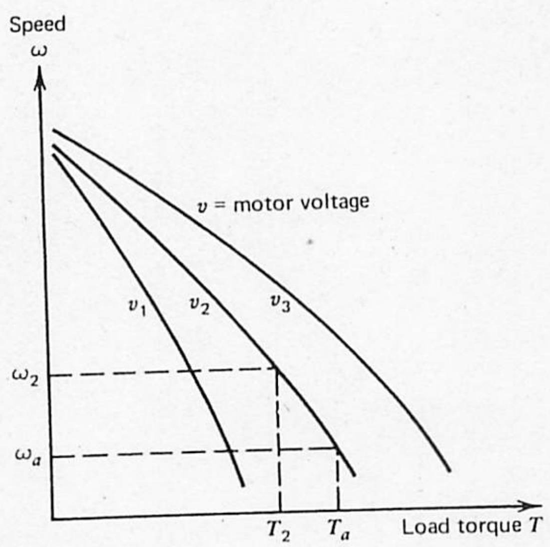


Figure 1.11 Steady-state operating curves for an electric motor.

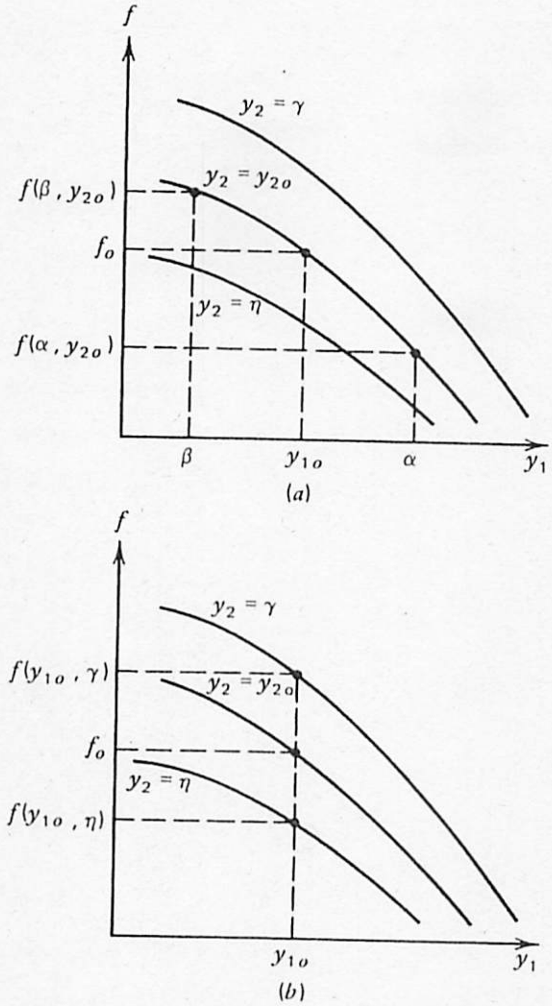


Figure 1.12 **Graphical computation of the linearization derivatives for the function $f(y_1, y_2)$ near the point $f_o = f(y_{1o}, y_{2o})$.**

$$\left(\frac{\partial f}{\partial y_1}\right)_o \cong \frac{f(\alpha, y_{2o}) - f(\beta, y_{2o})}{\alpha - \beta} \tag{1.4-22}$$

where the y_1 range (β, α) straddles y_{1o}. Similarly

$$\left(\frac{\partial f}{\partial y_2}\right)_o \cong \frac{f(y_{1o}, \gamma) - f(y_{1o}, \eta)}{\gamma - \eta} \tag{1.4-23}$$

The smaller $(\alpha - \beta)$ and $(\gamma - \eta)$ are made, the better the approximation, but this is limited by the ability to read the plot accurately for smaller increments.

1.5 FEEDBACK

A feature found in many static systems and in almost every dynamic system is one or more *feedback loops*, such as shown in Figures 1.2, 1.3, and 1.4. *Feedback* is the process by which an element's input is altered by its output. It occurs frequently in physiological and ecological systems and is deliberately employed in manufactured systems for several reasons, as we shall see. The human body utilizes many feedback loops for such purposes as body temperature control, blood pressure control, hand-eye coordination, and so forth. In nature, we can describe the alternating abundance and scarcity of prey and predators as being due to a feedback mechanism that prevents both species from becoming extinct.

Measurement of room temperature by a thermostat constitutes a feedback process because the measurement is used to influence the room temperature (Figure 1.3). Similarly the spring in Figure 1.2 acts as a feedback element. The greater the mass displacement, the greater the spring's restoring force attempting to return the mass to its rest position. The action of the diaphragm in the pressure regulator (Figure 1.4) combines the actions of a comparator and a sensor. As the pressure p increases, the diaphragm motion acts to move the valve to decrease the pressure. Thus the output (the pressure) is made to influence itself.

Control systems rely heavily on the properties of feedback. We now explore these properties in more detail.

Feedback Improves Linearity

As we indicated in Section 1.3, linear systems models will be the chief model form to be used in our study. One of the reasons for this is that the use of feedback often improves the linearity of the system. We can construct a system of elements whose individual behavior is nonlinear, but with proper use of feedback the resulting system's behavior will be approximately linear.

To illustrate this effect consider the nonlinear element shown in Figure 1.13*a*. Its input-output relation is

$$y = x^2 \tag{1.5-1}$$

If we introduce a feedback loop as in Figure 1.13*b*, we can write the following relations.

$$y = e^2$$

$$e = x - y$$

Thus

$$y = (x - y)^2 \tag{1.5-2}$$

The plot of y versus x for the original element and for the feedback system is shown in Figure 1.13*c*. The feedback system's input-output relation is closer to being a straight line and therefore is approximately linear over a wider range of x than for the original nonlinear element. This wider range is what we mean by "improved linearity."

Feedback improves linearity in dynamic as well as static systems. Chapter Six provides several examples of this effect.

Feedback Improves Robustness

The coefficient values and the form of a model are always approximations to reality and thus have some uncertainty associated with them. In addition, factors such as wear, heat, and pressure can cause the performance of a system to change with time.

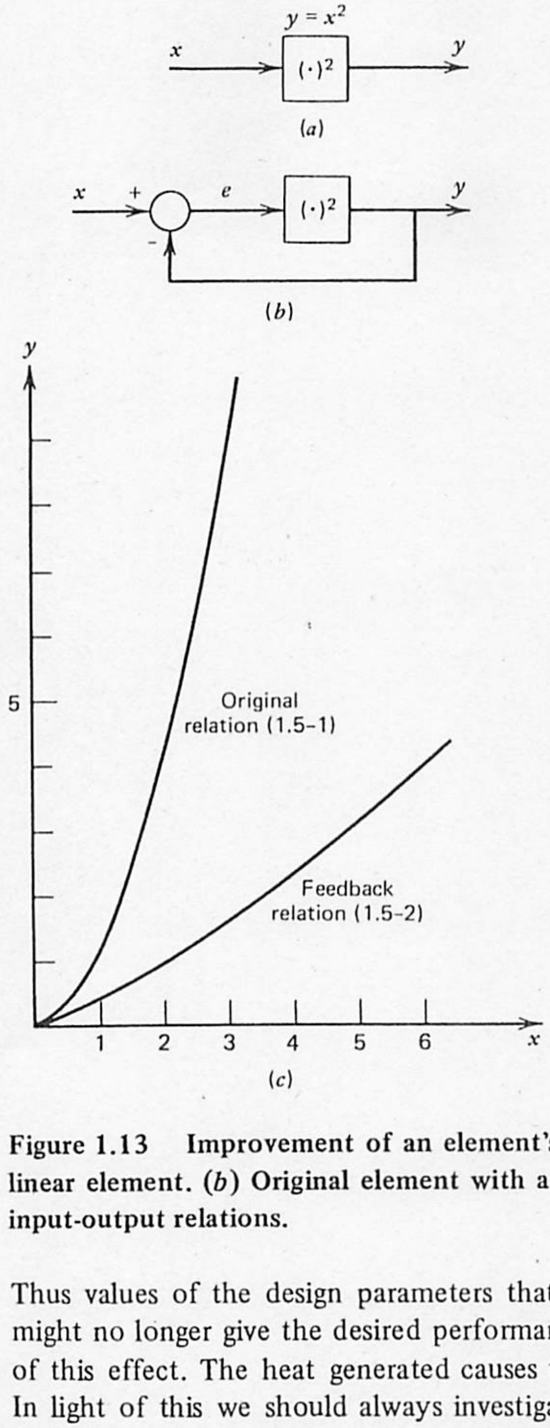


Figure 1.13 Improvement of an element's linearity with feedback. (*a*) Original nonlinear element. (*b*) Original element with a feedback loop added. (*c*) Plots of the two input-output relations.

Thus values of the design parameters that were optimal when the system was built might no longer give the desired performance. A vacuum-tube amplifier is an example of this effect. The heat generated causes the amplification factor to change in time. In light of this we should always investigate a prospective design to assess the sensitivity of its performance to uncertainties or variations in the system's parameters. Feedback can be used to improve the system's behavior in this respect.

We have seen that one purpose of the thermostat is to compensate for changes produced by variations in the outdoor environment – the system's "disturbances." This is another use for feedback, and systems that can maintain the output near its desired value in the presence of disturbances are said to have good *disturbance rejection.*

A system that has both good disturbance rejection and low sensitivity to parameter variations is said to be *robust*. We now illustrate how feedback can create or improve robustness.

Parameter Sensitivity

First consider the reduction of parameter sensitivity. The element shown in Figure 1.14*a* has a proportional constant G (called the *gain*). We wish the gain value to be $G = 10$ and select an element that has this nominal value. However, suppose that due to heat, wear, or poor construction, the actual value of G can vary by ± 10%. In this case the input-output relation will be somewhere between $y = 9x$ and $y = 11x$, and this is considered unacceptable.

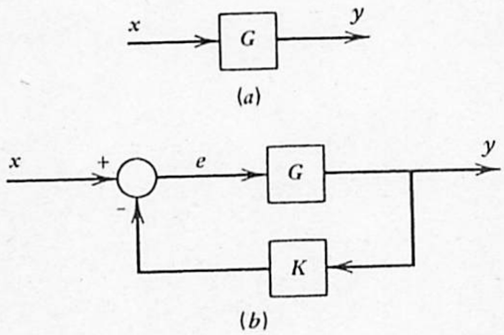


Figure 1.14 Reduction of parameter sensitivity with feedback. (*a*) Original element with an uncertain gain G. (*b*) Insertion of a feedback element with a reliable gain K. If $KG \gg 1$, $y \cong \frac{1}{K}x$.

To improve the situation we place a feedback loop with its own gain K around the basic element G. Presumably the physical element we select to produce the gain K will be relatively insensitive so that the value of K will be constant and predictable. This arrangement is shown in Figure 1.14*b*. The governing relations are

$$y = Ge, \quad e = x - Ky$$

or

$$y = \frac{G}{1 + GK}x \tag{1.5-3}$$

Note that if we pick G large enough so that $GK \gg 1$, then (1.5-3) becomes approximately

$$y \cong \frac{G}{GK}x = \frac{1}{K}x \tag{1.5-4}$$

The system's input-output relation becomes independent of G as long as GK is large! We now pick K to obtain our desired input-output relation, here $y = 10x$. Thus K must be $K = 0.1$, and we pick G such that $0.1G \gg 1$ or $G \gg 10$.

Let us use $G = 1000$ and see what happens. From (1.5-3)

$$y = \frac{1000}{1 + 100}x = 9.901x$$

which is very close to the desired relation. Now if G varies by ± 10% so that $G = 900$ and 1100, (1.5-3) gives

$$y = \frac{900}{1 + 90}x = 9.8901x$$

and

$$y = \frac{1100}{1 + 110}x = 9.91x$$

The sensitivity of the feedback system is much lower. This method of reducing the sensitivity is called *feedback compensation*.

A common engineering application of this approach is the stabilization of the gain of an electronic amplifier (Figure 1.15). Engineers working on this problem in the early part of the twentieth century discovered the principle of feedback compensation. The amplifier gain A is large but is subject to some uncertainty. The feedback loop is created by a resistor. Part of the voltage drop across the resistor is used to raise the ground level at the amplifier input. From the resistor's voltage-current relation we obtain

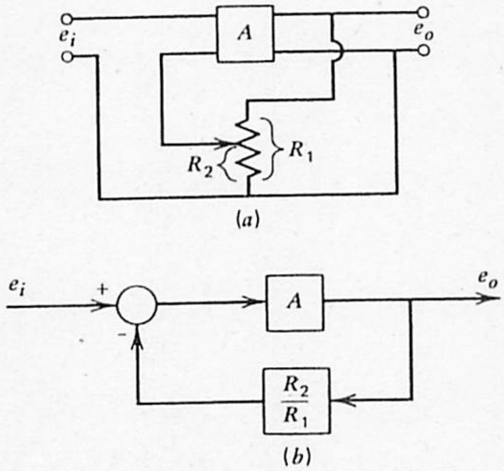


Figure 1.15 Feedback compensation of an amplifier. (*a*) Circuit diagram. (*b*) Block diagram.

$$e_o = A\left(e_i - e_o\frac{R_2}{R_1}\right) \tag{1.5-5}$$

or

$$e_o = \frac{A}{1 + A\dfrac{R_2}{R_1}}e_i$$

If $AR_2/R_1 \gg 1$, then

$$e_o \cong \frac{R_1}{R_2}e_i \tag{1.5-6}$$

Presumably the resistor values are sufficiently accurate and constant enough to allow the system gain R_1/R_2 to be reliable.

Another version of this application uses the *operational amplifier* (op amp). This is a voltage amplifier with a very large gain ($A = 10^5$ to 10^9) that draws a negligible current. If resistors are placed in series and parallel around the op amp, as shown in Figure 1.16, the system's input-output relation will be

$$e_o \cong -\frac{R_2}{R_1}e_i \tag{1.5-7}$$

This can be shown by writing the appropriate circuit equations. Since the current i_3 is negligible, the voltage e_1 is nearly zero, and $i_1 \cong i_2$. But

$$i_1 = \frac{e_i - e_1}{R_1}$$

$$i_2 = \frac{e_1 - e_o}{R_2}$$

Therefore

$$\frac{e_i - e_1}{R_1} = \frac{e_1 - e_o}{R_2}$$

Since $e_1 \cong 0$, it follows that

$$\frac{e_i}{R_1} \cong -\frac{e_o}{R_2}$$

which is equivalent to (1.5-7).

Op amps appear in many designs, and we will see more of them.

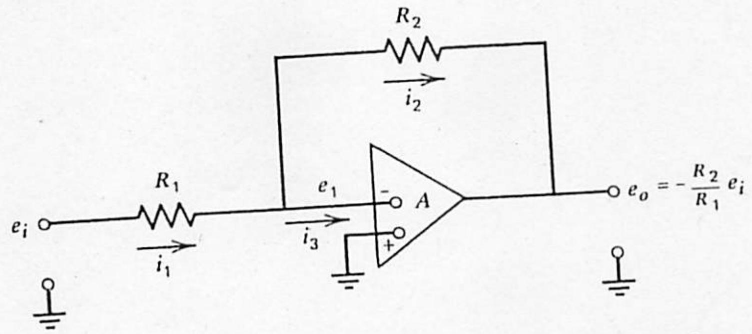


Figure 1.16 Op amp multiplier. Note the sign inversion.

Disturbance Rejection

Consider the system shown in Figure 1.17*a*. The disturbance is u, and the desired input-output relation between y and x is $y = 10x$. However, when $u \neq 0$, this relation is not obtained because

$$y = 5(2x - u) = 10x - 5u$$

This can be remedied by introducing a feedback loop and two gain elements, B and K, as shown in Figure 1.17*b*.

We can use the superposition principle to find the output y as a function of x and u. First set $u = 0$ and solve for y as a function of x.

$$y\Big|_{u=0} = \frac{10B}{1 + 10BK} x$$

Now replace u and set $x = 0$. Solve for y in terms of u to obtain

$$y\Big|_{x=0} = \frac{-5}{1 + 10BK} u$$

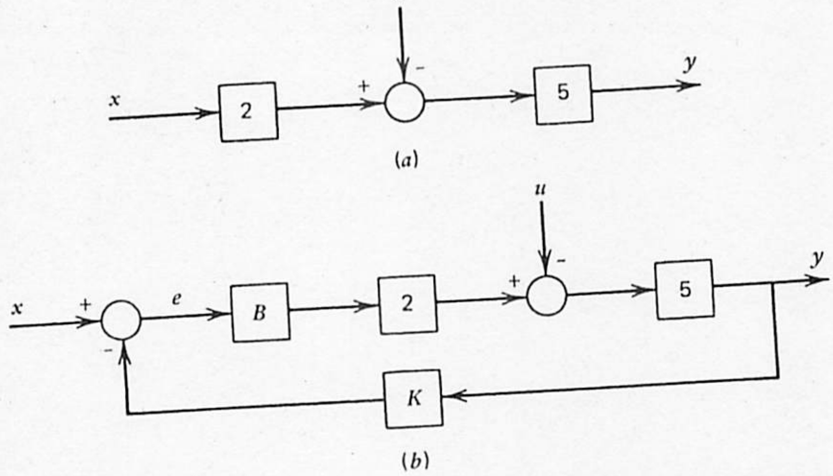


Figure 1.17 Use of feedback for disturbance rejection. (*a*) Original open-loop system. (*b*) Feedback system.

Invoking superposition we obtain the general expression for y by adding the above results.

$$y = \frac{10B}{1 + 10BK}x - \frac{5}{1 + 10BK}u \tag{1.5-8}$$

We desire that u have no effect on y, but this cannot be accomplished with finite values of B and K. Therefore we back off somewhat and require only that no more than 10% of the value of u show up in the output y; that is, we require that

$$\left|\frac{5}{1 + 10BK}u\right| \leq |0.1u|$$

This is satisfied if $5/(1 + 10BK) = 0.1$. From the desired relation $y = 10x$ we also have

$$\frac{10B}{1 + 10BK} = 10$$

The last two conditions give $B = 50$ and $K = 49/500$. This design meets the specifications and can be implemented if the values of B and K can be obtained physically and are reliable. The price paid for this improvement is that we have a more expensive system (due to the added elements B and K) and a possibly less reliable system (since there are more elements to fail).

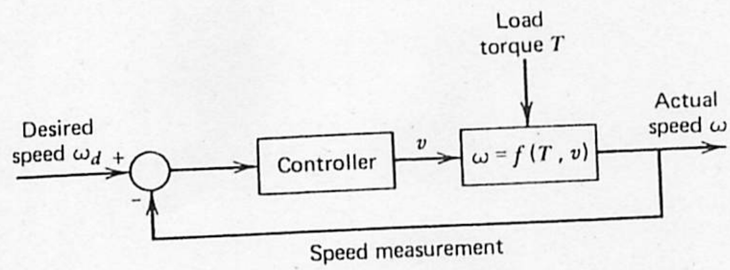
The Importance of Dynamic Models for Feedback Systems

We have used static systems to illustrate the properties of feedback, but its most important applications occur in dynamic systems. Consider the motor operating curves shown in Figure 1.11. Suppose that the motor speed, voltage, and load torque are ω_2, v_2, and T_2 initially, and that we wish to maintain the speed at ω_2. If the load torque increases, the speed will decrease. However, the operating curves do not tell us *how long* it will take for the new speed to be established. The curves represent only the steady-state behavior of the system as a static element. The inertia of the motor obviously prevents the speed from changing instantaneously from the value ω_2 to its new steady-state value.

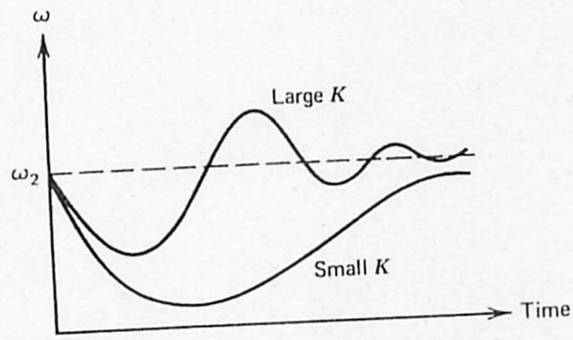
In order to keep the speed near its desired value we would use a controller. A block diagram of the general situation is shown in Figure 1.18*a*. The controller would sense that the speed had decreased and would increase the motor voltage. Again, the curves give no information about the time it would take for the speed to return to its desired value ω_2.

At this point it is not clear how we should design the controller to act. For example, should it change the voltage in proportion to the difference between the desired and actual speeds? Or should the voltage change be proportional to the *rate* of change of the speed difference? The fact that the motor and its load have inertia suggests that the time behavior of the speed depends to a great extent on the characteristics of the controller. For example, suppose we make the voltage change proportional to the *integral* of the speed difference such that

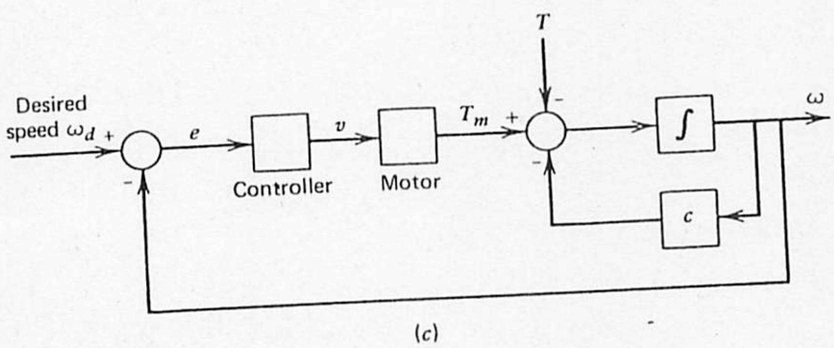
$$v = v_2 + K\int_0^t e\,dt \tag{1.5-9}$$



(a)



(b)



(c)

Figure 1.18 Motor speed control. (a) General block diagram. (b) Transient behavior of speed for different values of the controller gain K. (c) Block diagram using a dynamic model of the load.

where e is the speed error $\omega_2 - \omega$. We will see in Chapter Six that no steady-state error due to a constant load-torque disturbance will exist if this control scheme is used. This is because the voltage will not stop increasing until the speed difference e becomes zero. However, if the proportionality constant K is made too large, the controller can "overcompensate." The result is an oscillation in speed about the desired value. This effect is shown in Figure 1.18b. The initial time in that plot is the time at which the load torque changes. On the other hand, if K is made smaller, the oscillation does not occur, but the time to return ω to ω_2 might be very long.

Obviously we will need a dynamic model that includes the effect of the inertia in order to design the controller properly. Such a model can be obtained by applying Newton's law. We assume that the motor and load inertias can be lumped into one inertia I. The motor torque is T_m, and this is resisted by a friction torque T_f, which can often be modeled as proportional to the speed; that is, $T_f = c\omega$. By summing the torques on the inertia we obtain the dynamic model for the load speed.

$$I\frac{d\omega}{dt} = T_m - T - c\omega \tag{1.5-10}$$

The block diagram of the system with the controller and motor is shown in Figure 1.18*c*. When used with a motor model that relates motor torque to motor voltage, the preceding equation is often sufficient to design the controller – for example, to pick the proper value of K in (1.5-9).

The linearizing property of the feedback loop in the controller often allows us to model the system as a linear one. In addition, the reduction of the system's parameter sensitivity means that a lumped-parameter, low-order dynamic model is often satisfactory for the purpose of designing feedback systems. The model usually cannot be static, but must describe at least the dominant dynamic behavior of the system. We take up procedures for developing such models in the next chapter.

PROBLEMS

1.1 What is the causal relation for the following elements with the given inputs and outputs?

(a) A capacitor (charge as input; voltage as output).

(b) An inertia (torque as input; angular acceleration as output).

(c) Angular acceleration as input; angular velocity as output.

(d) A water tank with vertical sides (water volume as input; water height as output).

(e) The heat energy stored in a body as input; the body temperature as output.

1.2 Draw a block diagram for the following models. The inputs are u and v; the output is y. The variables x and e are internal variables. Show these on the diagram.

(a)
$$y = 5x$$
$$x = v + 3e - 4y$$
$$e = u - 2y$$

(b)
$$\dot{y} = 6x$$
$$\dot{e} = u - 2e$$
$$x = 5v + 3y + e$$

1.3 Obtain the input-output relations for each of the diagrams shown in Figure P1.3 The inputs are u and v; the output is y.

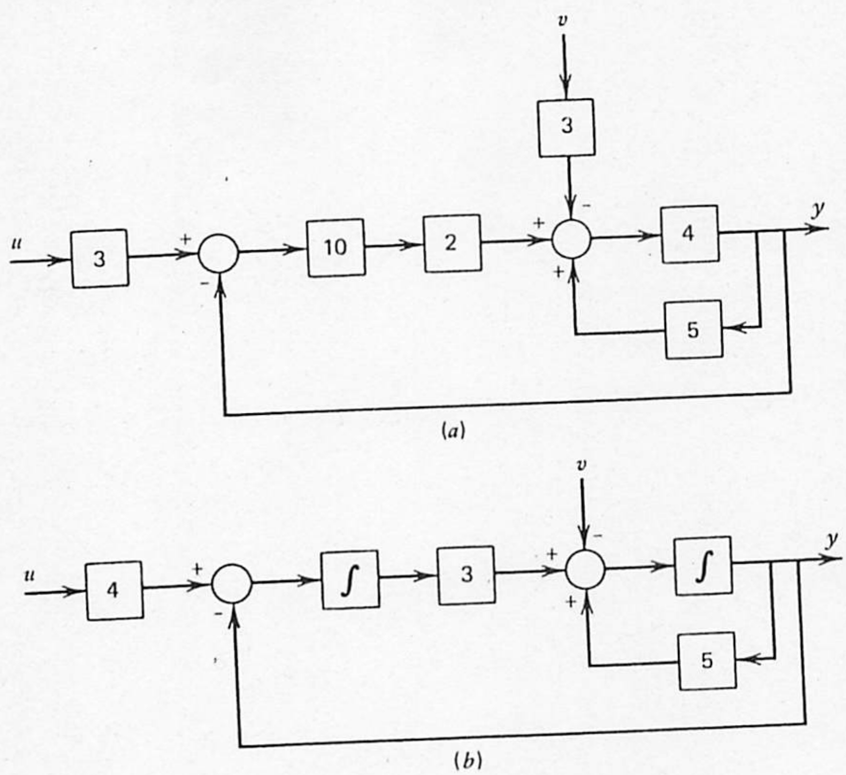


Figure P1.3

1.4 Water-level controllers represent the earliest examples of control systems. A simple version using a float and lever is shown in Figure P1.4.

(a) Discuss the system's operation. How can we adjust the water level that the system will maintain?

(b) Draw the block diagram with the desired level as the command input, the actual level as the output, and the change in water supply pressure as the disturbance.

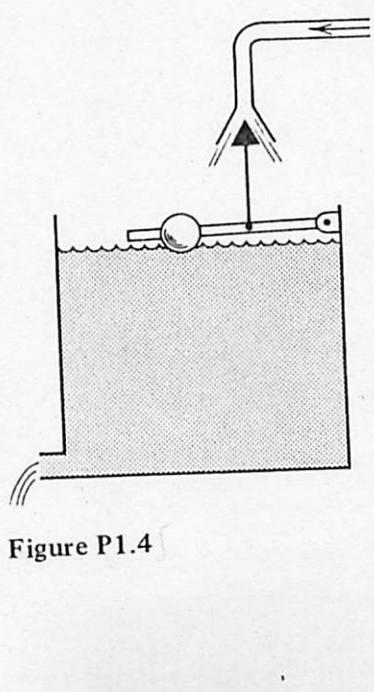


Figure P1.4

1.5 A person attempting to balance a stick on end in the palm of a hand constitutes a control system. Draw the block diagram of the system. Is more than one measurement involved?

1.6 A very long wire can have appreciable resistance. Should it be modeled as a distributed-parameter element?

1.7 A certain cantilever beam has considerable mass. Should it be modeled as a distributed-parameter element? Discuss how an approximate lumped-parameter model might be developed.

1.8 Is the following model nonlinear? Explain.

$$\frac{dx}{dt} = -3x + f, \qquad f = \begin{cases} +1 & \text{if } x \geqslant 0 \\ -1 & \text{if } x < 0 \end{cases}$$

1.9 Consider the savings growth model (1.3-7).

(a) Suppose that $x(0) = \$1.00$. Find $x(5)$, the amount of money at the end of five years.

(b) Generalize the results of (a) to find an expression for $x(k)$ in terms of $x(0)$ for any integer k.

1.10 Obtain a linearized expression for the following functions, valid near the given reference values.

(a) $w = \cos y, \quad y_o = 0$

(b) $w = \cos y, \quad y_o = \pi/4$

(c) $w = e^{3y}, \quad y_o = 1$

(d) $w = y_1^2 \sin y_2, \quad y_{1o} = 1, \quad y_{2o} = \pi/4$

(e) $w = y_1/y_2, \quad y_{1o} = 1, \quad y_{2o} = 3$

1.11 The area A of a rectangle is $A = y_1 y_2$, where y_1 and y_2 are the lengths of the sides.

(a) Obtain a linearized expression for A if $y_{1o} = 2, y_{2o} = 5$.

(b) Give a geometric interpretation of the error in the linearization.

1.12 Use the curves shown in Figure P1.12 to obtain a linearized expression for $w = f(y_1, y_2)$, valid near the point $y_{1o} = 1, y_{2o} = 10$.

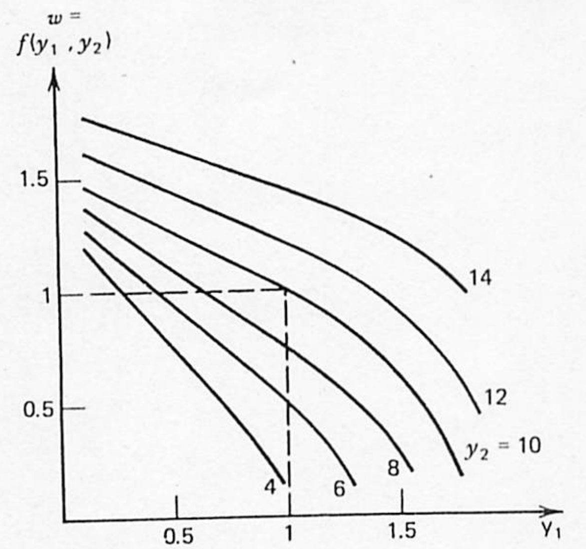


Figure P1.12

1.13 (a) Show that the two systems in Figure P1.13 have the same input-output relation when the gain K is 100.
(b) If the gain K is subject to a ± 10% uncertainty, which system is the *least* sensitive?

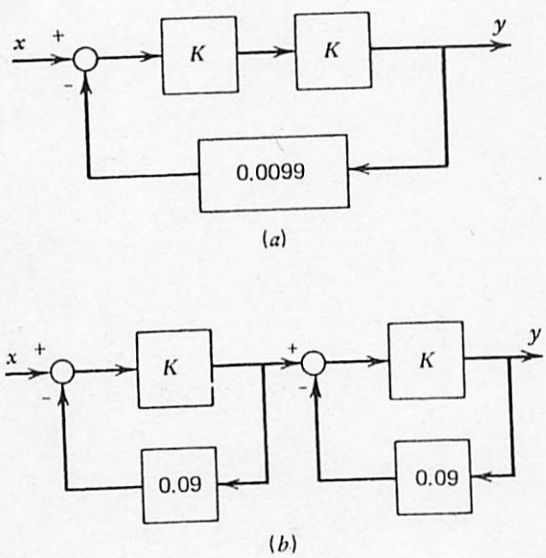


Figure P1.13

1.14 The following are common examples of control systems. Discuss their operation. Do they employ feedback?

(a) Toaster.
(b) Washing machine.
(c) Engine camshaft.
(d) Engine cooling system.
(e) Carburetor.
(f) Traffic light.

1.15 Explain the feedback action in the following systems.

(a) The law of supply and demand in economic systems.
(b) Temperature control in the human body.
(c) Predator-prey interactions.

1.16 This story circulates in several forms. A worker at a factory was in charge of activating the noon whistle at the plant. Being very conscientious and trusting, every day he phoned the research lab at the nearby university to set his clock according to their time. Eventually he became curious and asked the university researcher how he managed to keep his clock accurate. "Why, we set our clock by the noon whistle at the local factory" came the reply.

Is this a feedback process? If the university clock gains 2 minutes per day, what is the error in the timing of the "noon" whistle at the end of 5 days?